On Energy Distribution of Two Space-times with Planar and Cylindrical Symmetries

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Abstract

Considering encouraging Virbhadra's results about energy distribution of non-static spherically symmetric metrics in Kerr-Schild class, it would be interesting to study some space-times with other symmetries. Using different energy-momentum complexes, i.e. Møller, Einstein, and Tolman, in static plane-symmetric and cylindrically symmetric solutions of Einstein-Maxwell equations in 3+1 dimensions, energy (due to matter and fields including gravity) distribution is studied. Energy expressions are obtained finite and well-defined. calculations show interesting coincidences between the results obtained by Einstein and Tolamn prescriptions. Our results support the Cooperstock hypothesis about localized energy.

1   Introduction

Since 1916 when Einstein formulated his general theory of relativity, one of the greatest unsolved puzzles in the realm of the theory has been concept of gravitational energy. All that is said about equivalency of frames and arbitrariness of choosing coordinates fails here. Although, Einstein had believed that energy is localized in GR and introduced the first energy-momentum prescription, but, there is no general agreed definition of energy in GR, yet.

There are three main viewpoints about localization of energy: localization, non-localization, and quasi-loacalization. Misner et al. [1] argued that to look for a local energy-momentum is looking for the right answer to the wrong question. He showed that the energy can be localized only in systems which have spherical symmetry. Cooperstock and Sarracino [2] proved that if energy is localizable for spherical systems, then it can be localized in any system. In 1990, Bondi [3] argued that a non-localizable form of energy is not allowed in GR. Some physicists propose a new concept in this regard: *quasi-localization* (for example see [4]). Unlike energy-momentum prescriptions theory, quasi-localization theory does not restrict one to use particular coordinate system, but this theory have also its drawbacks [5].

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Besides Einstein [6] prescription, many other energy-momentum prescriptions was suggested by different persons that most well-known ones was given by Møller [6], Landau-Lifshitz [7], Papapetrou [8], Bergmann [9], Tolman [10], and Weinberg [11]. The idea of energy-momentum prescriptions was criticized for some reasons. First, using different energy-momentum prescriptions could lead to different energy distributions for same space-time. Second, except a few of them (like Møller prescription) for other prescriptions all calculations must be done in Cartesian coordinate system. Third, they are non-tensorial (pseudo-tensor) and thus their physical interpretation seems obscure. However, in 1999, Chang et al. [12] showed that every energy-momentum complex can be associated with distinct boundary term which gives the quasi local energy-momentum. By this way, he dispels doubts expressed about the physical meaning of energy-momentum complexes. Finally, we can define conserved angular momentum quantity only for symmetric prescriptions [11], while only a few of them are symmetric. In fact, anti-symmetric characteristic of Einstein's prescription was the main motivation for Landau and Lifshitz to look for an alternative prescription which is symmetric.

Gravitation becomes absent in local inertial frames and various selections of coordinate systems give different results for gravitational energy. In spite of serious objections to the root of the subject [1] there are many attempts to extract and study the common features of the different proposed energy-momentum prescriptions. Among these works most cases discussed were dealt with spherically symmetric metrics, pioneering by Virbhadra [13] who showed that for a general non-static spherically symmetric metric of Kerr-Schild class several energy-momentum complexes give the same energy distribution. Considering a general non-static spherically symmetric space-time of the Kerr-Schild class, he found a strong coincidence between different energy-momentum complexes in Kerr-Schild Cartesian coordinate system. Following this approach, it seems other types of symmetry including cylindrically and plane symmetries have been taken of less degree of consideration. Main contribution of Rosen and Virbhadra [?] to study cylindrical gravitational waves should be addressed.

Now in the frame work of just doing comparison between different energy prescriptions for gravitational fields, here we study the energy distribution of black plane (Plane-symmetric solution of Einstein-Maxwell equations) and black string (Cylindrically symmetric solution of Einstein-Maxwell equations) space-times. The remainder of the paper is organized as follows. In sections 2, we introduce the black plane and calculate the energy distribution of this space-time by Møller and Tolman prescriptions. We have compared these new results with energy distribution obtained by Einstein energy-momentum prescription. In section 3, we present the black string and explain about structure of this space-time, briefly. Using Møller, Einstein, and Tolman energy-momentum prescriptions we have calculated the energy distribution of this cylindrically symmetric space-time. In the final section we summarize the results and present our conclusions.

Conventions: we use geometrized units in which the speed of light in vacuum c and the Newtonian gravitational constant G are taken to be equal to 1. Through the paper Greek and Latin indices take values 0..3 and 1..3 respectively.

2   Black Plane Solution

The line element in general metric of static plane symmetry is defined as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(dx^2 + dy^2) \tag{1}$$

Cai and Zhang [16] started with the following action

$$S = \frac{1}{16\pi} \int_V d^4x \sqrt{-g}(R + 6\alpha^2 - F^{\mu\nu}F_{\mu\nu}) - \frac{1}{8\pi} \int_{\partial V} d^3x \sqrt{-h}K, \tag{2}$$

where R is the scalar curvature, $F_{\mu\nu}$ is Maxwell field, and $\alpha^2 = -\frac{\Lambda}{3} > 0$ presents the negative cosmological constant. The quantity h is the induced metric on ∂V, and K its extrinsic curvature. Equations of motion can be obtained by varying the action (2) as follows.

$$
\begin{aligned}
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 8\pi T_{\mu\nu}^{EM} + 3\alpha^2 g_{\mu\nu}, \\
0 &= \partial_\mu(\sqrt{-g}F^{\mu\nu}), \\
F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} &= 0,
\end{aligned}
\tag{3}
$$

where

$$T_{\mu\nu}^{EM} = \frac{1}{4\pi}(F_{\mu\lambda}F_\nu^\lambda - \frac{1}{4}g_{\mu\nu}F^2), \tag{4}$$

is the energy-momentum tensor of the Maxwell field. In the metric (1) solving Eqs. (3), Cai and Zhang found $A(r)$, $B(r)$, $C(r)$, and $F_{\mu\nu}$. Finally, they give the line element of this solution, black plane, as [16]

$$ds^2 = -(\alpha^2 r^2 - \frac{m}{r} + \frac{q^2}{r^2})dt^2 + (\alpha^2 r^2 - \frac{m}{r} + \frac{q^2}{r^2})^{-1}dr^2 + \alpha^2 r^2(dx^2 + dy^2) \tag{5}$$

where $\alpha^2 = -\frac{\Lambda}{3}$. m and q are obtained by using Gauss theorem and Euclidean action method of black membranes respectively [15] respectively as

$$m = -\frac{12\pi M}{\Lambda}, \quad q = 2\pi Q, \tag{6}$$

It should be noted that in (5) we have taken $r = |z|$ because of the reflection symmetry with respect to the $z = 0$ plane. Q, M, and Λ are electric charge density, ADM mass density, and negative cosmological constant, respectively. This solution is asymptotically anti-de Sitter not only in the transverse directions, but also in the membrane directions [16]. Calculating scaler curvature invariants it is not difficult to show that space-time (5) has a singularity at $r = 0$ plane. This singularity is enclosed by four event horizons. Causal structure of black

Table 1: Energy-momentum Prescriptions

Prescription	Energy-momentum Density	Energy-momentum
Møller [6]	$M_i^k = \frac{1}{8\pi}\chi_{i,l}^{kl}$ $\chi_i^{kl} = \sqrt{-g}(\frac{\partial g_{ip}}{\partial x^q} - \frac{\partial g_{iq}}{\partial x^p})g^{kq}g^{lp}$	$P_i = \int\int\int M_i^0 dx^1 dx^2 dx^3$ $= \frac{1}{8\pi}\int\int \chi_i^{0a} n_a ds$
Einstein [6]	$\Theta_i^k = \frac{1}{16\pi}H_{i,l}^{kl}$ $H_i^{kl} = -H_i^{lk} = \frac{g_{in}}{\sqrt{-g}}[-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}$	$P_i = \int\int\int \Theta_i^0 dx^1 dx^2 dx^3$ $= \frac{1}{16\pi}\int\int H_i^{0a} n_a ds$
Tolman [10]	$T_i^k = \frac{1}{8\pi}U_{i,l}^{kl}$ $U_i^{kl} = \sqrt{-g}(-g^{pk}V_{ip}^l + \frac{1}{2}g_i^k g^{pm}V_{pm}^l)$ $V_{jk}^i = -\Gamma_{jk}^i + \frac{1}{2}g_j^i \Gamma_{mk}^m + \frac{1}{2}g_k^i \Gamma_{mj}^m$	$P_i = \int\int\int T_i^0 dx^1 dx^2 dx^3$ $= \frac{1}{8\pi}\int\int U_i^{0a} n_a ds$

plane space-time Eq.(5) is similar to that of Reissner Nordström black holes [1]. In vacuum background ($M = Q = 0$) line element of (5) reduces to

$$ds^2 = -(\alpha^2 r^2)dt^2 + (\alpha^2 r^2)^{-1}dr^2 + \alpha^2 r^2(dx^2 + dy^2) \tag{7}$$

which is just the plane anti-de Sitter space-time.

2.1 Energy Distribution

Energy-momentum densities, and energy-momentum 4-vectors in Møller, Einstein, and Tolman prescriptions are presented in Table 1 briefly. Interested reader can refer to the mentioned references for details. We refer to this table for doing our calculations.

Halpern [17] used *Einstein prescription* and found energy distribution of black planes as:

$$E_E = M + \frac{\pi \Lambda Q^2}{3r} - \frac{\Lambda^2 r^3}{36\pi} \tag{8}$$

Considering this result, it would be interesting to study energy distribution of this space-time by other prescriptions. Here, we extend his work to Møller and Tolman prescriptions.

Using *Møller prescription* (Table 1) in space-time (5), components of χ_i^{jk} are obtained as

$$\chi_t^{tr} = \frac{\Lambda(2\Lambda r^4 - 3mr + 6q^2)}{9r} \tag{9}$$

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[1] Reissner Nordström anti-de Sitter solution and its energy distribution is reviewed in the Appendix 1

Other components of χ_i^{jk} are equal to zero. Energy-momentum components are found by surface integral presented in Table 1. It should be noted that for black planes, we choose the surface of integration to be a planar shell with fixed r (and then two fixed values of z). So, surface element, ds, is $ds = dxdy$ and μ_σ, is unit radial vector. After integration over planer shell with described ds and μ_σ, we find energy density as

$$E_M = P_0 = \frac{\chi_t^{tr}}{8\pi} \tag{10}$$

Substituting χ_t^{tr} from Eq.(9), energy density in black plane space-time (in Møller prescription) is obtained finite and well-define as

$$E_M = \frac{M}{2} + \frac{\Lambda \pi Q^2}{3r} + \frac{\Lambda^2 r^3}{36\pi} \tag{11}$$

Considering *Tolman prescription* (Teble 1), we calculate components of super potentials U_i^{kl} and obtain that only non-zero component is

$$U_t^{tz} = -\frac{2\Lambda(\Lambda r^4 + 3mr - 3q^2)}{9r} \tag{12}$$

We use planar shell with fixed r again as the surface of integration and obtain

$$E_T = P_0 = \frac{U_t^{tz}}{8\pi} \tag{13}$$

Substituting U_t^{tz} from Eq.(12) to Eq.(13), energy density is given by

$$E_T = M + \frac{\Lambda \pi Q^2}{3r} - \frac{\Lambda^2 r^3}{36\pi} \tag{14}$$

which is equal to energy density obtained by Einstein prescription in the same space-time (Eq.(8)). Radinschi [20] have obtained same coincidence between Einstein and Tolman energy-momentum prescriptions for Reissner Nordström anti-de Sitter space-time (Appendix). In vacuum background ($Q = \Lambda = 0$) or even when just $\Lambda = 0$ Eqs.(11),(14) reduce to $E = \frac{M}{2}$ and $E = M$ respectively.

Cai and Zhang [16] found that black plane solution has two horizons in each side of the plane $z = 0$. They defined an extremal case with only one horizon that occurs if

$$Q = \frac{\sqrt{3}M^{\frac{2}{3}}(-\Lambda)^{\frac{1}{6}}}{2\pi^{\frac{1}{3}}} \tag{15}$$

This only horizon is located at

$$r = (9\pi M)^{\frac{1}{3}}(-\Lambda)^{-\frac{2}{3}} \tag{16}$$

Halpern [17] calculated the energy contained inside this horizon by Einstein prescription. In this case, from Eq.(14) and compare it with Eq.(8) it is obvious that Tolman prescription also give same result. But, studying Møller

prescription in this case with different energy distribution (Eq.(11)) would be interesting. Substituting Eqs.(15),(16) in Eq.(11) we obtain total energy contained inside the horizon as

$$E_{ext} = M(\frac{3}{4} - \frac{3^{\frac{1}{3}}\Lambda^2}{12})$$ (17)

that for small values of Λ leads to

$$E_{ext} = \frac{3}{4}M$$ (18)

However energy distribution in Møller prescription is different from Einstein and Tolman prescriptions in general, but total energy within the black plane's event horizon for the extremal case in all three prescriptions is same and equal to three-quarters of the black plane's mass parameter.

3   Black String Solution

Static Cylindrically symmetric solution of Einstein-Maxwell equations is given by following line element [16], [18]

$$
\begin{aligned}
ds^2 &= -(\alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2})dt^2 + (\alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2})^{-1}dr^2 \\
&\quad + r^2 d\theta^2 + \alpha^2 r^2 dz^2
\end{aligned}
$$ (19)

where $\alpha^2 = -\frac{\Lambda}{3}$ and $-\infty < t, z < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$. Q and M are the ADM mass and charge per unit length in the z direction, respectively. This space-time is asymptotically anti-de Sitter in the both of transverse and string directions. Black string space-time has a singularity at $r = 0$ which is enclosed by two horizons.

3.1   Energy Distribution

Because of cylindrical symmetry of this space-time we choose a cylindrical surface surrounding the length L from the string symmetrically with radius r, as the surface of integration. So, the infinitesimal surface element is $ds = rd\theta dz$ and normal unit vector in Cartesian coordinate system (t, x, y, z) is $\mu_\alpha = (0, \frac{x}{r}, \frac{y}{r}, 0)$. If we remain in polar cylindrical coordinate system (t, r, θ, z) we have $\mu_\alpha = (0, 1, 0, 0)$. As mentioned in section 1, all calculations in Einstein and Tolman prescription must be done in Cartesian coordinate system. In this coordinate system (considering $\theta = arctan(\frac{y}{x})$, and $r = \sqrt{x^2 + y^2}$) Eq.(19) transform to the following line element.

$$
\begin{aligned}
ds^2 &= -\xi(r)dt^2 + (\frac{x^2}{r^2\xi(r)} + \frac{y^2}{r^2})dx^2 + \frac{2xy}{r^2}(\frac{1}{\xi(r)} - 1)dxdy \\
&\quad + (\frac{y^2}{r^2\xi(r)} + \frac{x^2}{r^2})dy^2 + \alpha^2 r^2 dz^2
\end{aligned}
$$ (20)

6

where

$$\xi(r) = \alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2}, \quad \alpha^2 = -\frac{\Lambda}{3} \tag{21}$$

Using *Møller prescription* (Table 1) for line element (20) we obtain following needed components of χ_i^{jk}.

$$
\begin{aligned}
\chi_t^{tx} &= -2x \frac{\alpha^4 r^{\frac{5}{2}} + 2M\alpha r - 4Q^2\sqrt{r}}{\alpha r^2} \\
\chi_t^{ty} &= -2y \frac{\alpha^4 r^{\frac{5}{2}} + 2M\alpha r - 4Q^2\sqrt{r}}{\alpha r^2}
\end{aligned}
\tag{22}
$$

With Eqs.(22) and after surface integration over described cylindrical surface in Cartesian coordinate system we have

$$
\begin{aligned}
E_M &= \tfrac{1}{8\pi} \int_0^L \int_0^{2\pi} \chi_t^{t\sigma} \mu_\sigma r d\theta dz \\
&= \frac{-\alpha^4 r^4 - 2M\alpha r + 4Q^2}{2\alpha r} L
\end{aligned}
\tag{23}
$$

It should be noted that similar calculations in polar cylindrical coordinate system also lead to the same result as expected.

Using *Einstein* and *Tolman prescriptions* (Table 1) for line element (20), non-zero components of super potentials H_i^{jk}, and U_i^{jk} are obtained as

$$
\begin{aligned}
H_t^{tx} = 2U_t^{tx} &= \frac{3\alpha^4 r^4 - \alpha^2 r^2 - 12M\alpha r + 12Q^2}{\alpha r^3} x \\
H_t^{tx} = 2U_t^{ty} &= \frac{3\alpha^4 r^4 - \alpha^2 r^2 - 12M\alpha r + 12Q^2}{\alpha r^3} y
\end{aligned}
\tag{24}
$$

With Eq.(24), and after calculation of the surface integral over described cylindrical surface in Cartesian coordinate system, we reach to the same results for both Einstein and Tolman prescriptions as follows.

$$
\begin{aligned}
E_{E,T} &= \frac{1}{16\pi} \int_0^L \int_0^{2\pi} H_t^{t\sigma} \mu_\sigma r d\theta dz \\
&= \frac{1}{8\pi} \int_0^L \int_0^{2\pi} U_t^{t\sigma} \mu_\sigma r d\theta dz \\
&= \frac{3\alpha^4 r^4 - \alpha^2 r^2 - 12M\alpha r + 12Q^2}{8\alpha r} L
\end{aligned}
\tag{25}
$$

4 Conclusion

Considering black plane and black string space-times, with planar and cylindrically symmetries respectively, we have studied their energy distributions by using Einstein, Møller, and Tolman prescriptions. All used prescriptions lead to finite and well-defined expressions for energy. These reasonable obtained results also tend to support the Cooperstock hypothesis that the localized energy is zero for regions where the energy-momentum tensor vanishes.

In the each of black plane and black string space-times, Einstein and Tolman prescriptions give equal results that can be considered as an extension of Virbhadra's viewpoint that different energy-momentum prescriptions may provide some basis to define a unique quantity. Møller prescription give different results due to pseudo-tensorial nature of energy-momentum complexes.

In addition, using Møller and Tolman prescriptions, our calculations on total energy within the black plane's event horizon for the extremal case leads to interesting results which are in agreement with previous results in Einstein prescription. However for black plane space-time energy distribution in Møller prescription is different from Einstein and Tolman prescriptions in general, but total energy within the black plane's event horizon for the extremal case in all these prescriptions is same and equal to three-quarters of the black plane's mass parameter.

Appendix

RN-AdS Solution and its Energy Distribution

Since casual structure of black plane space-time is similar to that of Reissner-Nordström anti-de Sitter (RN-AdS) space-time, we are interested to study this space-time more. General form of this metric is given by this line element [19]

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + rd\Omega_b^2$$

where

$$N(r) = -s\frac{r^2}{\alpha^2} + b - \frac{2m}{r} + \frac{q^2}{r^2}$$

with $\alpha^2 = -\frac{3}{|\Lambda|}$, $s = \frac{|\Lambda|}{\Lambda}$ is the sign of Λ, and

$$d\Omega_b = \begin{cases} d\theta^2 + \sin^2\theta d\phi^2, & b = 1,\ s = \pm 1; \\ d\theta^2 + d\phi^2, & b = 0,\ s = -1; \\ d\theta^2 + \sinh^2\theta d\phi^2, & b = -1,\ s = -1. \end{cases}$$

For $b = 1$, above metric describes the RN-AdS black holes. The event horizon of the black hole has the 2-sphere topology S^2, and the topology of the space-time is $R^2 \times S^2$ [20]. Energy distribution of this metric in the case of $b = 1$ in Einstein and Tolman prescriptions are same and given by [20]

$$E_{E,T}(r) = M - \frac{Q^2}{2r} + \frac{1}{6}\Lambda r^3$$

but in Møller prescription we have [21]

$$E_M(r) = M - \frac{Q^2}{r} - 3\frac{r^3}{\Lambda}$$

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